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ISOMORPHIC FACTORIZATIONS VIII: BISECTABLE TREES

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A tree is called even if its line set can be partitioned into two isomorphic subforests; it is bisectable if these forests are trees. The problem of deciding whether a given tree is even is known (Graham and Robinson) to be NP-hard. That for bisectability is now shown to have a polynomial time algorithm. This result is contained in the proof of a theorem which shows that if a tree S is bisectable then there is a unique tree T that accomplishes the bipartition. With the help of the uniqueness of T and the observation that the bisection of S into two copies of T is unique up to isomorphism, we enumerate bisectable trees.

1. Introduction

An *isomorphic factorization* of a graph G into t parts is a partition of the set of lines $E(G)$ into spanning subgraphs $F_1, \ldots, F_t$ all isomorphic to some graph F. We then write $F \in G/t$ and $F|G$, and say that G is divisible by t denoted by $t|G$. When $2|G$ it is natural to define G as *even*. Then G is *odd* if it is not even. Thus a tree S is even if $E(S)$ has a factorization into two isomorphic forests. We call S *bisectable* if these are two isomorphic trees. The graph theoretic notation and terminology follows [2], except that for a graph with p points and q lines, the *order* is p and the *size* is q.

In the first paper in this series [7], we proved that for any integers $t, p > 1$, if the Divisibility Condition $t \left| \binom{p}{2} \right.$ holds, then K_p/t is not empty. The second paper [10] expressed various combinatorial designs as isomorphic factorizations. The third [8] investigated when the corresponding Divisibility Condition for a complete multipartite graph G, namely, $t|q(G)$, assured the existence of an isomorphic factorization of G into t parts. The fourth [5] considered isomorphic Ramsey numbers and the fifth [9] applied the Divisibility Condition to digraphs. The sixth [12] studied the

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automorphisms of G as related to the isomorphic factors while the seventh [14] looked into isomorphic factorizations of regular graphs and regular tournaments.

Our main result in this eighth paper of the series is that for each bisectable tree S there is a unique tree $T \in S/2$. The proof of this statement contains an algorithm for testing whether a given tree S is bisectable that is linear in time $q(S)$. This is in contrast to the conclusion of the next, ninth paper [1], that the decision problem for even trees is NP-hard. It is also mentioned in [1] that this is already known for the problem of factoring a tree into any two trees of equal size.

Caro and Schönheim [0] ask for which trees G and H does $G|H$ hold, obtaining precise answers for certain special G such as stars.

We conclude by showing that if S is bisectable into two copies of T then the bisection is unique up to isomorphism. This enables us to enumerate bisectable trees both exactly and asymptotically.

2. Necessary condition for oddity

As a tree S of order p has size $p-1$, p even implies S odd by the Divisibility Condition. When p is odd the Divisibility Condition is satisfied, but we may still have S odd. And in case S is even, $S/2$ may or may not contain a tree. These possibilities are illustrated by the four trees of order 7 shown in Figure 1.

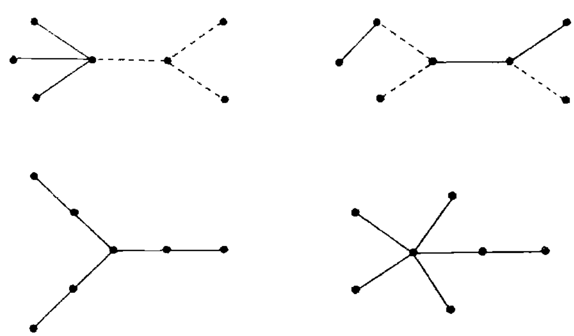

Fig. 1. Some trees of order 7

The first tree is shown factored into two copies of the tree $K_{1,3}$, with solid and dotted lines. Similarly the second tree is shown factored into two copies of $P_3 \cup P_2$. There is no factorization into two isomorphic trees, because in fact the only way to divide it into two trees of size 3 is for one to be a path and the other a star. Finally, the other two trees are the only odd ones among the 11 trees of order 7. Of the 47 trees of order 9, exactly nine are odd and are shown in Figure 2.

The oddity of some of these trees is apparent due to the existence of a point of large odd degree. For instance in Figure 1, the last tree has a single point of degree 5, while all other points have degree 1 or 2. Assume S is even and divisible into two copies of a forest F. Then F must contain a point of degree at least 3, in order that two copies should contain a point of degree 5. On the other hand the two points of degree 3 cannot be identified with the same point of S, as S contains no point of degree 6 or more. And they cannot be identified with distinct points of S, since there is no second point of degree 3 or more. It is easy to generalize this line of reasoning to prove the following statement.

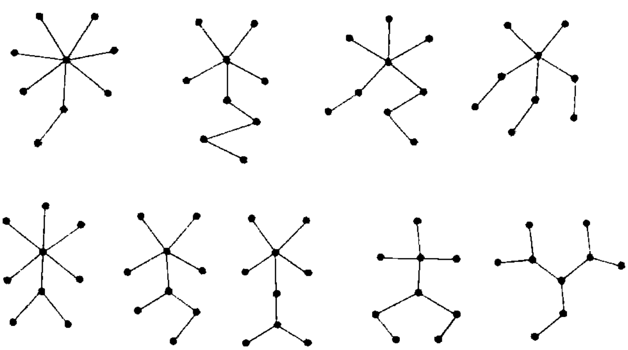

Fig. 2. The odd trees of order 9

Theorem 1. *Let $\Delta = \Delta(G)$ be the maximum degree. If $t \nmid \Delta$ and G has a unique point of maximum degree while all other points have degree less than Δ/t, then $t \nmid G$.* ▌

When the criteria of the theorem are applied to trees, we find that it accounts for just four of the odd trees of order 9, namely those shown in the top row of Figure 2. It is inevitable that such a sufficient condition for oddity will fall well short of being necessary. For the problem of determining which trees are odd is known to be NP-complete [1], while the condition of the theorem is readily determined in polynomial time.

There may be hope that oddity is easier to characterize for trees with bounded degrees. For instance, it is shown in [1] that with a single exception, every tree in which one point has degree 2 and the rest have degree 1 or 3 is even.

For divisibility into two isomorphic trees there is a polynomial time characterization, which is presented in the next section.

3. Criterion for bisectability

By contrast with the oddity of a tree S, the bisectability of S can be decided efficiently. That is, there is an algorithm with space and time bounds linear in the size of S for deciding its bisectability. We start by showing that for any t, a tree in S/t is unique. A linear algorithm for deciding the t-sectability of S is then deduced as a corollary. It is also shown as a corollary that for $t = 2$, the bisection itself is unique up to isomorphism.

Lemma 1. *If S is a t-sectable tree, then S and t alone determine the points of contact in any t-section of S and the weights of the branches of factors at points of contact.*

Proof. When $t=1$ there is no point of contact and nothing to prove. Proceeding by induction on t, suppose $t \geqq 2$. Let $T_1, \ldots, T_t$ be the factors of a t-section of S. Form the *factor-tree* of the t-section with point set consisting of the t factors and the points of contact, where a point of contact v is adjacent to a factor T_i whenever $v \in V(T_i)$. The factor-tree of any t-section is obviously a tree in which all the end-points are factors. Every nontrivial tree has at least two endpoints, so the factor-tree has at least two endpoints, called *endfactors*. Define the *boundary* of a tree as its endpoints if it is not trivial, and as its one point if it is. If we remove the end-factors from the factor-tree, the result is a tree (possibly trivial) in which the boundary points are points of contact. These we call *boundary points of contact*.

With a view to characterizing the boundary points of contact, let q be the size of each factor, so that the size of S is qt. Then it is straightforward that a point v of S is a boundary point of contact if and only if either

(1) there is exactly one branch of S at v having weight more than q and the total s of the weights of the other branches of S at v is at least q, or

(2) all of the branches of S at v have weight at most q.

In Case (2), v is the only point of contact, as other points of contact would have to lie on a branch at v with weight more than q, and there is no such branch. Consequently the branches of the various factors at v coincide with the branches of S at v.

In Case (1), let $0 \leqq r < q$ and $d \geqq 1$ be such that $s = qd + r$. The branches of S at v with size at most q all coincide with branches of factors at v. Let T_1 be the factor containing the line joining v to the rest of the largest branch at v. Then $q-r$ is the number of lines in T_1 on the largest branch at v, i.e., $q-r$ is the size of the corresponding branch of T_1 at v. This accounts for the sizes of all of the branches of factors at v. Now let $\hat{S}$ consist of the largest branch of S at v along with a star of r lines incident to v. The factors can be numbered so that $T_1, \ldots, T_{t-d}$ are those having lines in the large branch of S at v. Then $T_1, \ldots, T_{t-d}$ would be a $(t-d)$-section of the tree $\hat{S}$ except for the structure of T_1, which may not match the r-star introduced at v. However v is not a point of contact in a $(t-d)$-section of $\hat{S}$ and so the structure of $\hat{S}$ at v is irrelevant to the applicability of the induction hypothesis to $\hat{S}$.

The induction step for $t \geqq 2$ can now be summarized as follows. There must be at least one boundary point of contact in any t-section of S. The characterization, of which (1) and (2) are the cases, allows such a point v to be located using only S and t, since the latter determine q. Then the sizes of the branches of the factors at v are determined as above. If v satisfies (2), the remaining points of contact and the sizes of branches of factors at them are determined by $(t-d)$ and $\hat{S}$. ▮

Theorem 2. *For each tree S and $t \geqq 1$, there is at most one tree in S/t.*

Proof. When $t=1$, S/t consists of S alone. For $t \geqq 2$ we induct on t. As before, let q be the size of S divided by t. If q is not integral then of course S/t is empty, so suppose q is a positive integer. Now apply the algorithm in the proof of Lemma 1 to S and t. If some step of that algorithm cannot be completed, then of course S/t is empty and we are done. So suppose that the algorithm has been carried to completion. The result is a set C of points and a set W of weights with the property that if S has a t-section then C is the set of contact points and W is the set of weights of branches of the various factors at the points of C which they contain.

Let b be the maximum weight in W, so that $1 \leqq b \leqq q$. Let a *light branch* of a tree be one having size $q-b$ or less. We claim that the set of light branches of S is precisely the union of the sets of light branches of the t factor-trees of any t-section of S. One inclusion is obvious, because any branch of S having weight at most q is a branch of a single factor in any t-section of S. Conversely, if in some t-section of S there is a branch B of a factor-tree T at a point u such that B is not a branch of S, then B must contain a point $v \neq u$ which is in C. Consequently B has weight more than $q-b$, for if not, the branch of T at v containing u would have weight more than b, contrary to the maximality of b.

Let a *maximal light branch* of a tree be a light branch which is not properly contained in any other light branch. Note that in S each light branch is contained in a unique maximal one. In fact any two different maximal light branches of S must be line disjoint. For if both contain some line uv then one would contain all points of S which are, say, closer to u than to v. Since the other branch is not contained in the first, it would contain some point closer to v that to u and hence would contain all such points. Thus S would be included in the union of the two branches. But that contradicts the fact that S has weight at least $2q$ while each light branch has weight less than q.

The fact that maximal light branches of S are line disjoint implies the observation that if u, v are the roots of two such branches then no maximal light branch contains any line on the $u-v$ path in S. Let $\hat{S}$ be the minimal subgraph of S which contains the root points of all of the maximal light branches and all lines not contained in some such branch. Then our observation shows that $\hat{S}$ is a tree. In addition, suppose that $T_1, \ldots, T_t$ is a t-section of S and $\hat{T}_i$ is the intersection of T_i and S for $i=1, \ldots, t$. Then $\hat{T}_1, \ldots, \hat{T}_t$ is a t-section of $\hat{S}$ if the latter is nontrivial. For, as we saw, every light branch of T_i is a light branch of S, so $T_i \cong T_j$ implies $\hat{T}_i \cong \hat{T}_j$.

Suppose first that $\hat{S}$ is trivial, consisting of a single point v. Then $C=\{v\}$ and W consists of the weights of the branches of S at v. Also v is the only root point for maximal light branches. Since $\hat{S}$ contains no lines, all of the branches of S at v must be light. Thus $b \leqq q-b$, so no branch of S at v has weight more than $q/2$. As q is the weight of each factor in any t-section of S and each factor must contain v, it follows that v must be the centroid of each factor. Let $B_1, \ldots, B_k$ be the isomorphism types of the maximal light branches considered as rooted trees, with $m_1, \ldots, m_k$ as their respective multiplicities. In the case under consideration, S is t-sectable if and only if m_i/t is an integer for all $i=1, \ldots, k$. For if a tree T is to lie in S/t it must consist of m_i/t copies of B_i for $i=1, \ldots, k$ all joined at the centroid.

Assume now that $\hat{S}$ is nontrivial. The condition that m_i/t is integral for $i=1, \ldots, k$ is still necessary for the t-sectability of S. This is because the number of copies of B_i is the same in each of the t isomorphic factors of any t-section of S, and as we saw earlier the copies which occur as maximal light branches of S are precisely the union of those occurring as maximal light branches of the t separate factors. In the remainder of the proof it will be assumed that m_i/t is integral for $i=1, \ldots, k$.

Now let $t\hat{q}$ be the size of the nontrivial tree $\hat{S}$. Thus for any t-section $T_1, \ldots, T_t$ of S, its intersection $\hat{T}_1, \ldots, \hat{T}_t$ with $\hat{S}$ is a t-section of $\hat{S}$ in which each factor has size $\hat{q}$. In such a t-section of S let T_1, say, contain a contact point v such that a branch B of T_1 at v has the maximum possible size b. The other branches of T_1 at v are all light, since their total weight is $q-b$. Thus $\hat{T}_1$ has v as an endpoint, and so the branch of $\hat{S}$ at v containing $\hat{T}_1$ has weight $r\hat{q}$ for some integer r such that $0<r<t$.

It can now be assumed that v is a point of $\hat{S}$ such that some branch $\hat{B}$ of $\hat{S}$ at v has weight $r\hat{q}$ for integral r with $0<r<t$. For if not then $\hat{S}$ is not t-sectable, and hence S is not t-sectable. Clearly any t-section of $\hat{S}$ must consist of an r-section of $\hat{B}$ along with a $(t-r)$-section of the tree $\hat{S}-E(\hat{B})$. Let B be the branch of S at v containing $\hat{B}$. If a t-section of $\hat{S}$ is extendable to a t-section of S, then it must be possible to select a set A of light branches of S at v so that $A\cup B$ contains exactly rm_i/t copies of B_i as maximal light branches for $i=1, \ldots, k$. In fact, if n_i denotes the number of copies of B_i in $B-E(\hat{B})$ then A simply consists of $(rm_i/t)-n_i$ copies of B_i for $i=1, \ldots, k$. Thus in order for S to be t-sectable it is necessary that $n_i \leq rm_i/t$ and that at least $(rm_i/t)-n_i$ copies of B_i occur as branches of S at v, for $i=1, \ldots, k$. Assuming that these conditions are satisfied, S is t-sectable if and only if $A\cup B$ is r-sectable with factors isomorphic to some tree T and $S-E(A)-E(B)$ is $(t-r)$-sectable with factors isomorphic to the same tree T.

This concludes the inductive proof of the theorem, since $r<t$ and $t-r<t$ in the last case. ∎

Corollary 1. *For each t, there is an algorithm with time and space bounds linear in the size of S for deciding whether S/t contains a tree, and if so finding it.*

Proof. To implement the algorithm described in the proof of Theorem 2 in linear time, a procedure for testing isomorphism of trees in linear time is needed. One such procedure has been described by Hopcroft and Tarjan [11, pp. 140—142]. It is based on a method of assigning a canonical linear ordering to the neighbors of each point, proceeding recursively from the endpoints. This also allows the t-sectability question in case $\hat{S}$ is trivial to be settled in linear time. For then if the branches of S at the unique point of contact v are $D_1, \ldots, D_m$ in the canonical order at v, necessary and sufficient conditions for S to be t-sectable are $t|m$ and $D_i \cong D_{i+1}$ for $1 \leq i < m$ and $t \nmid i$. ∎

Corollary 2. *If a tree S is bisectable then its isomorphic factorization into two trees is unique up to isomorphism.*

Proof. In a bisection of S there is a single point v of contact, which is unique as seen in Lemma 1. We can proceed by induction on the size of S.

If $\hat{S}$ is trivial, the factorization was seen in the proof of Theorem 2 to be unique, whatever the value of t. When $\hat{S}$ is nontrivial, its size is less than that of S and so by induction the bisection of $\hat{S}$ is unique up to isomorphism. It was also seen in the proof of Theorem 2 that any bisection of S extends the bisection of $\hat{S}$. The only uncertainly in extending the bisection of $\hat{S}$ to a bisection of S is in the division of the maximal light branches at v between the two factors. However the numbers of such branches in each factor are determined by S, and so the extension to a factorization of $\hat{S}$ is unique up to isomorphism. ∎

In the next section this corollary is used as the basis of an enumeration of bisectable trees. However the uniqueness of bisections relies on having just one reduction step in the algorithm. For $t>2$ the factorization of a tree S into t copies of a tree T need not be unique up to isomorphism. This is illustrated in Figure 3 for $t=3$ (trisection) and in Figure 4 for $t=4$ (tetrasection).

It is an open question whether the t-sectability of a tree S can be decided by an algorithm linear in the size of S, independent of t.

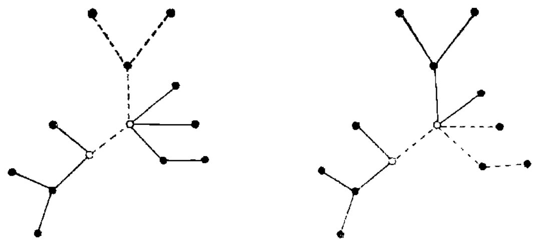

Fig. 3. Inequivalent trisections of a tree

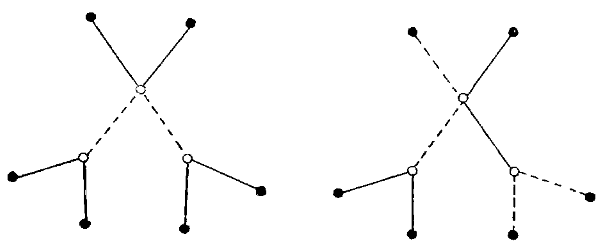

Fig. 4. Inequivalent tetrasections of a tree

4. Counting bisectable trees

Standard methods, as in [3, 6, and 13], can be applied to enumerate bisectable trees on the basis of Lemma 2 of the previous section. Therefore the derivation of recurrence equations for the exact number and asymptotic formulas for the behavior as $p \to \infty$ is given in outline form. Numerical results for $p \leqq 100$ were obtained, but are not included in the present paper. They are available on request from the second author.

Two points are *similar* if some automorphism maps one to the other, that is, if they are in the same orbit of the point automorphism group. The number of dissimilar points is precisely the number of nonisomorphic rooted versions of any graph. Let $T_{p,k}$ denote the number of rooted trees of order p having exactly k dissimilar points and let $t_{p,k}$ be the corresponding number for unrooted trees.

We now show that the number f_p of trees of order $2p-1$ which are bisectable is given by

$$f_p = \sum_{k=1}^{p} \binom{k+1}{2} t_{p,k}. \tag{4.1}$$

For suppose we consider bisections of a tree into two copies of a tree A of order p which has k point orbits. Such a bisection is formed from two copies of A by identifying a point of one copy with a point of the other copy. Up to isomorphism the bisection so obtained is determined by the orbits of the points chosen to be identified, so there are $\binom{k+1}{2}$ different bisections into two copies of A. By Corollary 2 the correspondence between bisections and bisectable trees is 1 – 1, so we can sum $\binom{k+1}{2}$ over all nonisomorphic trees A of order p to obtain (4.1).

To determine $t_{p,k}$ and hence f_p it is convenient to use the ordinary generating functions

$$T(x, y) = \sum_{p=1}^{\infty} \sum_{k=1}^{p} T_{p,k} x^p y^k,$$

$$t(x, y) = \sum_{p=1}^{\infty} \sum_{k=1}^{p} t_{p,k} x^p y^k.$$

As usual, a recurrence for the rooted trees is obtained by removing the root point and considering the unordered sequence of rooted trees obtained by deleting the original root and rooting the points which were adjacent to it. In this sequence the possible contributions of copies of any particular rooted tree R of order p with k orbits has as its ordinary generating function the series

$$1 + x^p y^k + x^{2p} y^k + x^{3p} y^k + \dots = 1 + \frac{x^p y^k}{1 - x^p}.$$

This is because any positive number of copies of R contributes just k point orbits. These contributions from different rooted trees are independent, since two points are similar only if they belong to isomorphic rooted trees in the sequence. In addition the original root point contributes a factor of xy since it is isomorphic to no other point. Thus we have

$$T(x, y) = xy \prod_{p=1}^{\infty} \prod_{k=1}^{p} \left(1 + \frac{x^p y^k}{1 - x^p}\right)^{T_{p,k}}. \tag{4.2}$$

This relation can be used directly to calculate the numbers $T_{p,k}$. However what will be required are the sums $N_p = \sum k T_{p,k}$ and $F_p = \sum \binom{k+1}{2} T_{p,k}$. The corresponding ordinary generating functions $N(x)$ and $F(x)$ can be expressed in terms of partial derivatives of $T(x, y)$ evaluated at $y = 1$. Specifically,

$$N(x) = T_y(x, 1) \quad \text{and} \quad F(x) = \frac{1}{2} T_{yy}(x, 1) + T_y(x, 1).$$

Of course $T(x, 1) = T(x)$ is the classical ordinary generating function for the numbers T_p of rooted trees of order p. Setting $y = 1$ in (4.2) immediately gives Cayley's expression for $T(x)$; see [3, equation 3.1.10].

Otter's recurrence for T_p is

$$T_p = \frac{1}{p-1} \sum_{k=1}^{p-1} T_{p-k} \sum_{d|k} d T_d, \tag{4.3}$$

for $p \geqq 2$ where $T_1 = 1$. This can be obtained from Cayley's relation for $T(x)$ by differentiating and comparing like coefficients of x; see [3, equation 3.1.9]. If we differentiate (4.2) with respect to y and set $y = 1$, we obtain

$$N(x) = T(x) + T(x) N(x). \tag{4.4}$$

In terms of the coefficients, this gives for $p \geqq 1$ the recurrence

$$N_p = T_p + \sum_{k=1}^{p-1} N_k T_{p-k}. \tag{4.5}$$

Differentiating (4.2) twice with respect to y, setting $y = 1$ and combining with (4.4) yields

$$F(x) = T(x)\left(F(x) - F(x^2) + \frac{1}{2} N(x^2)\right) + \frac{1}{2}\left(T(x) + N(x) + N(x)^2\right). \tag{4.6}$$

Comparing coefficients of x^p, we have

$$F_p = \sum_{k=1}^{p-1} T_{p-k}\left(F_k - F_{k/2} + \frac{1}{2} N_{k/2}\right) + \sum_{k=1}^{[(p-1)/2]} N_{p-k} N_k + \frac{1}{2}\left(T_p + N_p + N_{p/2}^2\right) \tag{4.7}$$

for $p \geqq 1$. It is understood that F_α or N_α is zero if α is not integral.

The relation between $t(x, y)$ and $T(x, y)$ is

$$t(x, y) = T(x, y) - \frac{1}{2}\left(T(x, y)^2 + T(x^2, y^2)\right) + T(x^2, y). \tag{4.8}$$

This is obtained from a slight variant of Otter's dissimilarity characteristic theorem, for trees [3, Equation 3.2.3]. The variant is proved in the same way as the original, with cognizance taken of the number of orbits of points at each step. The result is that the number of unrooted trees is expressed as the number of point-rooted trees, minus the number of line-rooted trees in which automorphisms are not allowed to reverse the root line, plus the number of trees containing a line of symmetry. The corresponding generating functions appear on the right side of (4.8).

From (4.1) it follows that the ordinary generating function $f(x)$ for bisectable trees of order $2p-1$ can be expressed as $\frac{1}{2} t_{yy}(x, 1) + t_y(x, 1)$. Applying the indicated operations to (4.8), we find

$$f(x) = F(x) - F(x^2) - T(x)F(x) - \frac{1}{2} N(x)^2 + \frac{1}{2} N(x^2).$$

Relation (4.6) can be used to simplify this, to

$$f(x) = \frac{1}{2}\left(N(x) + T(x) + (1 + T(x))(N(x^2) - 2F(x^2))\right). \tag{4.9}$$

The corresponding expression for f_p is

$$f_p = \frac{1}{2}\left(N_p + T_p + N_{p/2} - 2F_{p/2} + \sum_{k=1}^{[(p-1)/2]} (N_k - 2F_k) T_{p-2k}\right). \tag{4.10}$$

Relations (4.3), (4.5), (4.7) and (4.10) were used to compute the values of f_p for $p \leqq 100$.

The asymptotic behavior of f_p can be deduced from known facts about $T(x)$. Considering x as a complex variable, about $x = 0$, $T(x)$ has a positive radius of convergence η. The best value to date [4].for η is 0.338321856899208 .... Also η is the sole singularity of $T(x)$ on the circle of convergence, $T(\eta)=1$, and $|T(x)|<1$ for $|x|<\eta$ and for $x\neq\eta$ with $|x|=\eta$. Around $x=\eta$, $T(x)$ has the expression

$$T(x) = 1-b(\eta-x)^{1/2}\pm\ldots \tag{4.11}$$

in powers of $(\eta-x)^{1/2}$, where $b=2.6811281472677\ldots$.

Solving (4.4) for $N(x)$ gives

$$N(x) = T(x)/(1-T(x)). \tag{4.12}$$

Thus $N(x)$ also is analytic at all $|x|\leqq\eta$ except for $x=\eta$, about which $N(x)$ has the form

$$N(x) = \frac{1}{b}(\eta-x)^{-1/2}\pm\ldots. \tag{4.13}$$

Since $\eta^{1/2}>\eta$, $N(x^2)$ is analytic at all $|x|\leqq\eta$. In view of (4.6), then, $F(x)$ is analytic for all $|x|<\eta$, so that $F(x^2)$ is also analytic for all $|x|\leqq\eta$. From (4.9) it then follows that $f(x)$ is analytic at all $|x|\leqq\eta$ except $x=\eta$. Finally, from (4.11) and (4.13) there is a neighborhood of $x=\eta$ in which the expansion

$$f(x) = \frac{1}{2b}(\eta-x)^{-1/2}\pm\ldots \tag{4.14}$$

is valid, the remainder consisting of terms in non-negative powers of $(\eta-x)^{1/2}$.

Standard analytic techniques now can be applied to read off the asymptotic behavior of f_p; see [3, Section 9.5], [6], [13] or [4]. The result is

$$f_p = \frac{n^{-1/2}\eta^{-n}}{2b(\pi\eta)^{1/2}}\left(1+O\left(\frac{1}{n}\right)\right). \tag{4.15}$$

Thus the growth of f_p is dominated by the factor η^{-p}, where $\eta^{-1}=2.9557652856\ldots$. The constant factor $C=1/2b(\pi\eta)^{1/2}$ has the value $0.1808891292\ldots$. A comparison of the exact value of f_p with the approximation $a_p=Cn^{-1/2}\eta^{-n}$ for $p\leqq 100$ shows that the relative error multiplied by p tends to a constant value, as expected.

Let t_p be the number of unrooted trees of order p. The average number of orbits among all trees of order p is the ratio T_p/t_p, that is, the value of k averaged over these trees. The average value of $\binom{k+1}{2}$ is f_p/t_p. By using the known asymptotic behavior of T_p and t_p [3, Equations 9.5.29 and 9.5.36] it is easy to verify that

$$f_p/t_p \sim \binom{T_p/t_p}{2} \sim \frac{2n^2}{b^4\eta^2}.$$

The value of $2/b^4\eta^2$ is $0.33814237285\ldots$.

Finally, the proportion of trees of odd order $2p-1$ which are bisectable is f_p/t_{2p-1}. This ratio goes to zero exponentially since the growth of t_{2p-1} is dominated by the factor η^{-2p}.

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